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DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF Sp(p, q)

BY

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ABSTRACT

Let $p > q$ and let $G = \mathrm{Sp}(p,q)$. Let $P = LN$ be the maximal parabolic subgroup of G with Levi subgroup $L \cong \mathrm{GL}_q(\mathbb{H}) \times \mathrm{Sp}(p-q)$. For $s \in \mathbb{C}$ and μ a highest weight of $\mathrm{Sp}(p-q)$, let $\pi_{s,\mu}$ be the representation of P such that its restriction to N is trivial and $\pi_{s,\mu}|_L = \det_q^s \boxtimes \tau^\mu_{p-q}$, where $\det_q$ is the determinant character of $\mathrm{GL}_q(\mathbb{H})$ and τ^μ_{p-q} is the irreducible representation of $\mathrm{Sp}(p-q)$ with highest weight μ. Let $I_{p,q}(s,\mu)$ be the Harish-Chandra module of the induced representation $\mathrm{Ind}_P^G \pi_{s,\mu}$. In this paper, we shall determine the module structure and unitarity of $I_{p,q}(s,\mu)$.

1. Introduction

1.1. Let $p > q$ and let $G = \mathrm{Sp}(p,q)$. Let $P = LN$ be the maximal parabolic subgroup of G with Levi subgroup $L \cong \mathrm{GL}_q(\mathbb{H}) \times \mathrm{Sp}(p-q)$. For $A \in \mathrm{GL}_q(\mathbb{H})$, let $\det_q(A)$ be the usual determinant of A realized as an element of $\mathrm{GL}_{2q}(\mathbb{C})$ (see pages 18–19 of [C]). Note that $\det_q A \in \mathbb{R}^+$ for all $A \in \mathrm{GL}_q(\mathbb{H})$. For $n \in \mathbb{Z}^+$, let

$$\Lambda^+(n) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \colon \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$$

denote the set of dominant weights of $\mathrm{Sp}(n)$. For each $\lambda \in \Lambda^+(n)$, τ_n^λ shall denote the finite dimensional irreducible representation of $\mathrm{Sp}(n)$ with highest weight λ.

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Now for $s \in \mathbb{C}$ and $\mu \in \Lambda^+(p-q)$, let $(\pi_{s,\mu}, U^{s,\mu})$ be the representation of P such that its restriction to N is trivial and

$$\pi_{s,\mu}|_L = \det_q^s \boxtimes \tau_{p-q}^{\mu}.$$

We form the normalized induced representation

(1)

$$\mathrm{Ind}_P^G \pi_{s,\mu} = \{f \colon G \to U^{s,\mu} \colon f \text{ is } C^\infty, f(gp) = (\delta(p^{-1}))^{\frac{1}{2}} \pi_{s,\mu}(p^{-1}) f(g), g \in G, p \in P\}$$

where δ is the modular function of P given by $\delta(a) = (\det_q a)^{2p+1}$, $a \in L$, and on which G acts by left translation. Let $I_{p,q}(s,\mu)$ be the Harish-Chandra module of $\mathrm{Ind}_P^G \pi_{s,\mu}$. In this paper, we shall (i) describe all the irreducible subquotients of $I_{p,q}(s,\mu)$, (ii) determine the module structure of $I_{p,q}(s,\mu)$, and (iii) determine the unitarity of $I_{p,q}(s,\mu)$ and all its irreducible subquotients.

Note that $I_{p,q}(s,\mu)$ is not multiplicity free when restricted to the maximal compact subgroup K of G.

1.2. We shall briefly describe our methods and our main results. We first note that the representation $I_{p,q}(s,\mu)$ can be embedded into a representation of the larger group $\widetilde{G} = \mathrm{Sp}(p,p)$ whose module structure is well known ([J], [S], [Zh]). More specifically, let $\widetilde{P} = \widetilde{L}\widetilde{N}$ be the maximal parabolic subgroup of $\widetilde{G}$ with its Levi subgroup $\widetilde{L} \cong \mathrm{GL}_p(\mathbb{H})$. We form the normalized induced representation $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}} \det_p^s$ (cf. (1)), and let $(\sigma_s, I_p(s))$ denote its Harish-Chandra module. We shall embed $I_{p,q}(s,\mu)$ into $I_p(s)$ in Proposition 2.5.1. Let $I_{p,q}(s,\mu)'$ denote its image in $I_p(s)$. Let

$$W_1 \subsetneq W_2 \subseteq I_p(s)$$

be infinitesimal $\mathrm{Sp}(p,p)$-submodules of $I_p(s)$ such that $R := W_2/W_1$ is an irreducible subquotient of $I_p(s)$. Define

$$R' := (W_2 \cap I_{p,q}(s,\mu)')/(W_1 \cap I_{p,q}(s,\mu)'). \tag{2}$$

Thus R' can be identified with a subquotient of $I_{p,q}(s,\mu)$.

Theorem 1.2.1: *Let R be an irreducible subquotient of $I_p(s)$ and $\mu \in \Lambda^+(p-q)$. Define R' as in (2). If R' is nonzero, then it is an irreducible subquotient of $I_{p,q}(s,\mu)$. Moreover, all irreducible subquotients of $I_{p,q}(s,\mu)$ are of this form.*

Using this theorem, we completely determine the module structure of $I_{p,q}(s,\mu)$ and the unitarity of all the irreducible subquotients of $I_{p,q}(s,\mu)$. The results are given in Theorems 3.3.1, 3.4.1 and 3.5.1. We also obtained the "long complementary series" in Corollary 3.5.2. In the case of the orthogonal group, this phenomenon was first observed by J-S Li ([Li]). It has also been shown in [LTZ] that representations in the long complementary series are the theta lifts of certain lowest weight modules.

1.3. The composition factors of the principal series of $\mathrm{Sp}(p,1)$ was determined by [BK] using Vogan's algorithm.

1.4. We will show in a future paper ([LL2]) that some of the subquotients we obtained are the theta lifts (and more generally Howe quotients) of lowest weight modules of $\mathrm{O}^*(2m)$. Indeed this fact provided the initial evidence that Theorem 1.2.1 holds.

1.5. In an earlier paper ([LL1]), we have determined the structure and unitarity of similar families of representations for the groups $\mathrm{U}(p,q)$ and $\mathrm{Spin}_0(p,q)$. In establishing the results for these cases, we have used a very crucial fact that the restriction of an irreducible representation of $\mathrm{U}(n)$ (resp. $\mathrm{Spin}(n)$) to $\mathrm{U}(n-1)$ (resp. $\mathrm{Spin}(n-1)$) is multiplicity free. However, this does not hold for representations of $\mathrm{Sp}(n)$. Thus we need substantially different and more involved arguments for the $\mathrm{Sp}(p,q)$ case.

1.6. This paper is arranged as follows. In §2, we shall summarize the module structure of $I_p(s)$ (cf. [J], [S], [Zh]) and embed $I_{p,q}(s,\mu)$ into $I_p(s)$. We state our theorems on the module structure and unitarity of $I_{p,q}(s,\mu)$ in §3. In §4, we review the results of Molev ([Mo]) on finite dimensional representations of $\mathrm{Sp}(n)$ and prove several lemmas for later use. In §5, we use Molev's result to construct bases for $I_p(s)$, $I_{p,q}(s,\mu)$ and their subquotients. Finally, we prove Theorem 1.2.1 in §6 and §7.

Lemma 5.2.1 describes the explicit action of the Lie algebras on a basis of the irreducible representation in Theorem 1.2.1. This will be useful for future computations.

2. Restriction of the degenerate series of $\mathrm{Sp}(p,p)$

2.1. Let $\widetilde{G} = \mathrm{Sp}(p,p)$. In this section, we shall review the module structure of the degenerate principal series representation $I_p(s)$ of $\widetilde{G}$ given in §1.2. By restricting the action of $\widetilde{G}$ to $\mathrm{Sp}(p,q) \times \mathrm{Sp}(p-q)$, we show that we can embed $I_{p,q}(s,\mu)$ into $I_p(s)$.

2.2. We note that $\widetilde{G}$ has a maximal compact subgroup $\widetilde{K} = \mathrm{Sp}(p) \times \mathrm{Sp}(p)$ and $I_p(s)$ has $\widetilde{K}$-types

$$I_p(s) = \sum_{\lambda \in \Lambda^+(p)} V_\lambda \tag{3}$$

where for each $\lambda \in \Lambda^+(p)$, $V_\lambda \cong \tau_p^\lambda \boxtimes \tau_p^\lambda$.

2.3. In order to state Theorem 2.4.1 below, we need to recall a definition. Let H be a Lie group and let V a Harish-Chandra module of H of finite length. A **module diagram** $\mathcal{G}(V)$ of V ([A]) is a directed simple graph where the vertex set of $\mathcal{G}$ is the set of all irreducible subquotients of V. There is a directed edge from the vertex R_2 to the vertex R_1 if and only if there are submodules $V_2 \subsetneq V_1 \subseteq V$ satisfying a non-split exact sequence

$$0 \to R_2 \to V_1/V_2 \to R_1 \to 0.$$

2.4. Let $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{k}}$ be the Lie algebras of $\widetilde{G}$ of $\widetilde{K}$ respectively, and let

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{p}}$$

be the corresponding Cartan decomposition. For each $\lambda \in \Lambda^+(p)$, let pr_λ : $I_p(s) \to V_\lambda$ be the canonical projection, and let m_λ: $\mathfrak{p}_\mathbb{C} \otimes V_\lambda \to I_p(s)$ be the $\widetilde{\mathfrak{k}}_\mathbb{C}$ map given by the Lie algebra action

$$m_\lambda(X \otimes v) = X(v) \quad (X \in \mathfrak{p}_\mathbb{C},\ v \in V_\lambda).$$

For $\lambda, \lambda' \in \Lambda^+(p)$, let $T_{\lambda,\lambda'} = \mathrm{pr}_{\lambda'} \circ m_\lambda$. By Schur's lemma, $T_{\lambda,\lambda'}$ is unique up to a nonzero scalar. If $\lambda' \neq \lambda \pm \varepsilon_i$ for all $1 \leq i \leq p$, then $T_{\lambda,\lambda'} = 0$, and (see [J])

(i) $T_{\lambda,\lambda+\varepsilon_i} \neq 0$ if and only if $\lambda_i \neq -s - p + i - \frac{3}{2}$,

(ii) $T_{\lambda,\lambda-\varepsilon_i} \neq 0$ if and only if $\lambda_i \neq s - p + i - \frac{1}{2}$.

This immediately leads to the following theorem on the structure of $I_p(s)$.

THEOREM 2.4.1 ([J], [S] and [Zh]): *Let $s \in \mathbb{C}$.*

(a) *$I_p(s)$ is reducible if and only if $s \in \frac{1}{2} + \mathbb{Z}$ and $|s| \geq \frac{3}{2}$, and it is unitarizable if and only if either $\mathrm{Re}(s) = 0$ (unitary induction) or $s \in (-\frac{3}{2}, \frac{3}{2})$ (complementary series).*

(b) *Suppose $s \in \frac{1}{2} + \mathbb{Z}$ and $|s| \geq \frac{3}{2}$. Set $l := \max(-|s| + p + \frac{1}{2}, 0)$. Then for each $l \leq t \leq p$, $I_p(s)$ has an irreducible subquotient $R_t(s)$ whose $\widetilde{K}$-types are*

$$R_t(s) = \sum \{V_\lambda \simeq \tau_p^\lambda \boxtimes \tau_p^\lambda \colon \lambda_t \geq |s| - p + t - 1/2 \geq \lambda_{t+1}\}. \tag{4}$$

The module diagram of $I_p(s)$ is given by

$$R_l(s) \longleftarrow R_{l+1}(s) \longleftarrow \cdots \longleftarrow R_p(s) \text{ if } s \leq -3/2,$$
$$R_l(s) \longrightarrow R_{l+1}(s) \longrightarrow \cdots \longrightarrow R_p(s) \text{ if } s \geq 3/2.$$

$R_t(s)$ is unitarizable if and only if either (i) $t = p$ *or* (ii) $\frac{3}{2} \leq |s| \leq p + \frac{1}{2}$ *and* $t = -|s| + p + \frac{1}{2}$.

For a Harish-Chandra module V, we shall denote its dual module by V^*. Then $R_t(s)^* \cong R_t(-s)$ as infinitesimal Sp(p,q)-modules.

2.5. Fix $1 \leq q < p$. Then $\mathrm{Sp}(p,q) \times \mathrm{Sp}(p-q) \subseteq \mathrm{Sp}(p,p)$. For each $\mu \in \Lambda^+(p-q)$, let $I_p(s)_\mu$ denote the τ^μ_{p-q}-isotypic component of $I_p(s)$.

PROPOSITION 2.5.1:

(i) *For any $s \in \mathbb{C}$ and $\mu \in \Lambda^+(p-q)$,*

$$I_p(s)_\mu \cong I_{p,q}(s,\mu) \boxtimes \tau^\mu_{p-q}.$$

Hence

$$I_p(s) \cong \sum_{\mu \in \Lambda^+(p-q)} I_{p,q}(s,\mu) \boxtimes \tau^\mu_{p-q}. \tag{5}$$

(ii) *The summands $I_{p,q}(s,\mu)$ on the right hand side of (5) have distinct infinitesimal characters.*

Proof: The proof for (i) is similar to the cases of U(p,q) and Spin$_0(p,q)$ ([LL1]). For (ii), we note that $I_{p,q}(s,\mu)$ has infinitesimal character

$$\left(s+q-\frac{1}{2}, s+q-\frac{3}{2}, \ldots, s-q+\frac{1}{2}, \mu_1+p-q, \mu_2+p-q-1, \ldots, \mu_{p-q}+1\right) \tag{6}$$

and this is uniquely determined up to the action of the Weyl group $(\pm 1)^{p+q} \rtimes S_{p+q}$. Hence given p, q and s, the infinitesimal character determines μ. This proves the proposition. ∎

2.6. Let $\mu \in \Lambda^+(p-q)$ and let v_μ be a fixed vector in τ^μ_{p-q}. Let $I_p(s)_{v_\mu}$ be the image of $I_{p,q}(s,\mu) \boxtimes v_\mu$ under the isomorphism given in Proposition 2.5.1. Then we can identify $I_{p,q}(s,\mu)$ with the subspace $I_p(s)_{v_\mu}$ in $I_p(s)$. Under the action of $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$,

$$I_{p,q}(s,\mu) \cong I_p(s)_{v_\mu} = \sum_{\lambda \in \Lambda^+(p,\mu)} \tau^\lambda_p \boxtimes (\tau^\lambda_p)_{v_\mu} \tag{7}$$

where

$$\Lambda^+(p,\mu) = \{\lambda \in \Lambda^+(p) \colon \lambda_i \geq \mu_i \geq \lambda_{i+2q},\ \forall 1 \leq i \leq p-q\}$$

and

$$(\tau_p^\lambda)_{v_\mu} = \mathrm{span}\{\phi(v_\mu) \in \tau_p^\lambda | \phi \in \mathrm{Hom}_{\mathrm{Sp}(p-q)}(\tau_{p-q}^\mu, \tau_p^\lambda)\}. \tag{8}$$

3. Module structure and unitarity

3.1. In this section, we shall assume Theorem 1.2.1 and we will apply it to obtain the module structure and unitarity of $I_{p,q}(s,\mu)$.

3.2. Let $\mu \in \Lambda^+(p-q)$ and let v_μ be a highest weight vector in τ_{p-q}^μ. By (7) we identify $I_{p,q}(s,\mu)$ with the subspace $I_p(s)_{v_\mu}$ in $I_p(s)$. Let R be an irreducible subquotient of $I_p(s)$. Then by Theorem 2.4.1, either $R = I_p(s)$ or $R = R_t(s)$ for some t. Let R_μ be the τ_{p-q}^μ-isotypic component in R and let R_{v_μ} be the space of $\mathrm{Sp}(p-q)$ highest weight vectors in R_μ. If $W_1 \subseteq W_2$ are infinitesimal $\mathrm{Sp}(p,p)$ submodules of $I_p(s)$ such that $R = W_2/W_1$, then

$$R_{v_\mu} \cong \frac{W_2 \cap I_p(s)_{v_\mu}}{W_1 \cap I_p(s)_{v_\mu}}.$$

Thus R_{v_μ} can be identified with a subquotient of $I_{p,q}(s,\mu)$. In fact, by Theorem 1.2.1, if $R_{v_\mu} \neq 0$, then it is an irreducible subquotient of $I_{p,q}(s,\mu)$. Moreover, all irreducible subquotients of $I_{p,q}(s,\mu)$ are of this form.

3.3. Module diagram. The following theorem describes the module diagram (see §2.3) of $I_{p,q}(s,\mu)$. Its proof will be given in §5.4.

Theorem 3.3.1: *If $s + \frac{1}{2} \in \mathbb{Z} - \{0,1\}$, $\mu \in \Lambda^+(p-q)$, and*

$$l_1 = \min\{j \colon \mu_{j+1} \leq |s| - 1/2 - p + j\}, \quad l_2 = \max\{j \colon \mu_{j-2q} \geq |s| - 1/2 - p + j\}.$$

Then the module diagram of $I_{p,q}(s,\mu)$ is given by

$$\begin{aligned}
&(R_{l_1}(s))_{v_\mu} \longleftarrow (R_{l_1+1}(s))_{v_\mu} \longleftarrow \cdots \longleftarrow (R_{l_2}(s))_{v_\mu} \text{ if } s \leq -3/2,\\
&(R_{l_1}(s))_{v_\mu} \longrightarrow (R_{l_1+1}(s))_{v_\mu} \longrightarrow \cdots \longrightarrow (R_{l_2}(s))_{v_\mu} \text{ if } s \geq 3/2.
\end{aligned}$$

Thus the module diagram of $I_{p,q}(s,\mu)$ is a spanning subgraph of $I_p(s)$.

3.4. The following theorem identifies those representations $I_{p,q}(s,\mu)$ which are irreducible.

THEOREM 3.4.1: *Let $s \in \mathbb{C}$ and $\mu \in \Lambda_0^+(p-q)$. Then $I_{p,q}(s,\mu)$ is irreducible if and only if one of the following conditions holds:*

(a) $s+\frac{1}{2} \in \mathbb{C}\backslash\mathbb{Z}$.

(b) $s = \pm\frac{1}{2}$.

(c) $p \geq 2q$, $s+\frac{1}{2} \in \mathbb{Z}$, $|s| \geq \frac{3}{2}$, $\max(2q, -|s|+p+\frac{1}{2}) \leq j \leq p$ *and*

$$\mu_j = \mu_{j-1} = \cdots = \mu_{j-2q+1} = |s| - 1/2 - p + j.$$

In particular, if $j > p-q$, then $\mu_j = 0$, so that the above condition for irreducibility is equivalent to $p \geq 2q$, $\mu_{j-2q+1} = 0$ and $|s| = p - j + \frac{1}{2}$.

Proof: By Theorem 2.4.1, if s satisfies either Condition (a) or (b), then $I_p(s)$ is irreducible. Thus in these cases, $I_{p,q}(s,\mu)$ is irreducible by Theorem 1.2.1. The representations $I_{p,q}(s,\mu)$ in part (c) are exactly those representations in Theorem 3.3.1 with $l_1 = l_2$, that is, their module diagrams consist of a single vertex. ∎

3.5. UNITARITY. Next we shall determine the unitarity of $I_{p,q}(s,\mu)$ and its irreducible subquotients using the method of [LL1]. We shall omit the proof and only outline the main ideas.

A subquotient of $I_{p,q}(s,\mu)$ is unitarizable if and only if it is isomorphic to its Hermitian dual. By comparing the infinitesimal characters using (6), this occurs only if s is real or purely imaginary. In the latter case $I_{p,q}(s,\mu)$ is irreducible and unitarily induced.

Next we assume that s is real and R is an irreducible subquotient of $I_p(s)$. One can construct a $\mathfrak{sp}(p,p)$-invariant form $\langle .,. \rangle$ (which need not be positive definite) on R and its signature on the $\tilde{K}$-type V_λ can be calculated. Write $R = R^+ \oplus R^-$ where R^+ (resp. R^-) is the sum of $\tilde{K}$-types of positive (resp. negative) signature. Since R is $\tilde{K}$-multiplicity free, it is easy to see that the restriction of $\langle .,. \rangle$ to R_{v_μ} is a nontrivial nondegenerate $\mathfrak{sp}(p,q)$-invariant Hermitian form $\langle .,. \rangle'$ on R_{v_μ}. Moreover, R_{v_μ} is irreducible, so $\langle .,. \rangle'$ is the unique form up to a scalar multiple. Hence R_{v_μ} is unitarizable if and only if it lies in R^+ or R^-. Checking the last condition is a tedious but straightforward computation. The results are summarized in the following theorem.

THEOREM 3.5.1: *Let $s \in \mathbb{C}$ and $\mu \in \Lambda^+(p)$.*

(A) (Unitarity of $I_{p,q}(s,\mu)$) *$I_{p,q}(s,\mu)$ is unitarizable if and only if one of the following holds:*

  (a) (Unitary induction) $\mathrm{Re}(s) = 0$.

  (b) (Restriction of the complementary series of Sp(p,p)) $s \in (-3/2, 3/2)$.

(c) $p \geq 2q+1$, $\mu_a = 0$ *for some* $1 \leq a \leq p-2q$, $s \in \mathbb{R}$, *and* $|s| < p-2q-a+\frac{5}{2}$.

(B) (Unitary subquotients) *Suppose that* $s+\frac{1}{2} \in \mathbb{Z} - \{0,1\}$, *so that* $I_p(s)$ *is reducible.*

(a) (Restriction of the unitarizable subquotients in $I_p(s)$)

(i) *If* $\mu_{p-2q} \geq |s| - \frac{1}{2}$, *then* $(R_p(s))_{v_\mu}$ *is unitarizable.*

(ii) *If* $\frac{3}{2} \leq |s| \leq p+\frac{1}{2}$, $l = -|s|+p+\frac{1}{2}$ *and* $\mu_{l+1} = 0$, *then* $(R_l(s))_{v_\mu}$ *is unitarizable.*

(b) (New unitary subquotients) *If* $p \geq 2q+1$, $2q \leq j < p$, $\mu_{j-2q} \geq |s| - p + j - \frac{1}{2}$ *and* $\mu_{j-2q+1} = 0$, *then* $(R_i(s))_{v_\mu}$ *is unitarizable.*

(c) *Parts* (a) *and* (b) *give all the unitarizable subquotients of* $I_{p,q}(s,\mu)$.

The following corollary, which is a consequence of Theorem 3.5.1(A)(c), gives the "long complementary series."

COROLLARY 3.5.2: *If* $p \geq 2q$, $s \in \mathbb{R}$ *and* $|s| < p-2q+\frac{3}{2}$, *then* $I_{p,q}(s,0)$ *is unitarizable.*

4. A basis of representations of $\mathfrak{sp}(p)$

4.1. For any positive integer p, we shall write $\mathfrak{sp}(p)$ for $\mathfrak{sp}(p,\mathbb{C})$. In this section, we shall review the results of Molev ([Mo]) on the structure of finite dimensional irreducible representations of $\mathfrak{sp}(p)$ and prove several lemmas for later use.

Molev enumerates the rows and columns of $2p \times 2p$ complex matrices by the indices $-p, -p+1, \ldots, -1, 1, \ldots, p$. Let $\{E_{ij}\colon i,j = -p,\ldots,-1,1,\ldots,p\}$ be the standard basis of $\mathfrak{gl}(2p,\mathbb{C})$. Let

$$F_{ij} = E_{ij} - \theta_{ij} E_{-j,-i}$$

where $\theta_{ij} = \operatorname{sgn} i \cdot \operatorname{sgn} j$. Then

$$\mathfrak{sp}(p) = \operatorname{span}\{F_{ij}\colon i,j = -p,-p+1,\ldots,-1,1,\ldots,p\}.$$

For $r < p$, we shall identify $\mathfrak{sp}(r)$ with the subalgebra of $\mathfrak{sp}(p)$ spanned by $\{F_{ij}\colon i,j = -r+1,\ldots,-1,1,\ldots,r-1\}$. In this way, we obtain a chain of subalgebras

$$\mathfrak{sp}(1) \subseteq \mathfrak{sp}(2) \subseteq \cdots \subseteq \mathfrak{sp}(p-1) \subseteq \mathfrak{sp}(p).$$

Let $\lambda = (\lambda_1,\ldots,\lambda_p) \in \Lambda^+(p)$ and let τ_p^λ denote the irreducible representation of $\mathfrak{sp}(p)$ with highest weight λ. By restricting τ_p^λ to the above chain of subalgebras, Molev constructs a basis for τ_p^λ which is analogous to the Gelfand–Zetlin bases for finite dimensional irreducible representations for $\mathfrak{u}(p)$ and $\mathfrak{so}(p)$.

4.2. DEFINITION. Let $\lambda \in \Lambda^+(p)$. A pattern $M = (m_{ij}, m'_{ij})$ of $\mathfrak{sp}(p)$ associated with λ is an array of integers of the form

$$\begin{array}{cccccccccc}
m_{p1} & & m_{p2} & & \cdots & & \cdots & & m_{pp} & \\
& m'_{p1} & & m'_{p2} & & \cdots & & \cdots & & m'_{pp} \\
& & m_{p-1,1} & & \cdots & & \cdots & & m_{p-1,p-1} & \\
& & & m'_{p-1,1} & & \cdots & & \cdots & & m'_{p-1,p-1} \\
& & & & & & & & \cdots & \\
& & & & & & & & \cdots & \\
& & & & & & & & m_{11} & \\
& & & & & & & & & m'_{11}
\end{array}$$

where $(m_{p1}, m_{p2}, \ldots, m_{pp}) = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ and the following inequalities hold:

$$m_{k1} \geq m'_{k1} \geq m_{k2} \geq m'_{k2} \geq \cdots \geq m_{kk} \geq m'_{kk} \geq 0,$$
$$m'_{l1} \geq m_{l-1,1} \geq m'_{l2} \geq m_{l-1,2} \geq \cdots \geq m_{l-1,l-1} \geq m'_{ll} \geq 0,$$

for $k = 1, \ldots, p$ and $l = 2, \ldots, p$.

Our definition differs from that given in page 595 of [Mo]. In his notation, $\lambda_{ij} = -m_{i,i-j+1}$ and $\lambda'_{ij} = -m'_{i,i-j+1}$.

4.3. NOTATION. (i) For $\lambda \in \Lambda^+(p)$, define the pattern $M_\lambda = (m_{ij}, m'_{ij})$ by $m_{ij} = m'_{ij} = \lambda_j$ for $1 \leq j \leq i \leq p$. Then ζ_{M_λ} is a highest weight vector in τ_p^λ.
(ii) Let $M = (m_{ij}, m'_{ij})$ be a pattern of $\mathfrak{sp}(p)$. Let $1 \leq r \leq p$ and we define $d_r(M)$ to be the pattern of $\mathfrak{sp}(r)$ obtained from M by deleting its top $2p - 2r$ rows. We also define

$$\mathbf{r}_i(M) := (m_{i1}, \ldots, m_{ii}), \quad \mathbf{r}'_i(M) := (m'_{i1}, \ldots, m'_{ii}),$$
$$|\mathbf{r}_i(M)| := \sum_{j=1}^{i} m_{ij}, \quad |\mathbf{r}'_i(M)| := \sum_{j=1}^{i} m'_{ij}.$$

PROPOSITION 4.3.1 ([Mo], Thm 1.1): *τ_p^λ has a basis*

$$\mathcal{B}(\lambda) = \{\zeta_M\}$$

parameterized by all patterns M associated with λ such that for $1 \leq k \leq p$,

$$F_{kk}\zeta_M = (|\mathbf{r}_k(M)| + |\mathbf{r}_{k-1}(M)| - 2|\mathbf{r}'_k(M)|)\zeta_M, \tag{9}$$

$$F_{k,-k}\zeta_M = \sum_{i=1}^{k} A_{ki}(M)\zeta_{M-\delta'_{ki}}, \tag{10}$$

$$F_{-k,k}\zeta_M = \sum_{i=1}^{k} B_{ki}(M)\zeta_{M+\delta'_{ki}}, \tag{11}$$

$$F_{k-1,-k}\zeta_M = \sum_{i=1}^{k-1} C_{ki}(M)\zeta_{M+\delta_{k-1,i}} + \sum_{i=1}^{k}\sum_{j,m=1}^{k-1} D_{kijm}(M)\zeta_{M-\delta'_{ki}-\delta_{k-1,j}-\delta'_{k-1,m}}, \tag{12}$$

where $A_{ki}(M)$, $B_{ki}(M)$, $C_{ki}(M)$ and $D_{kijm}(M)$ are **nonzero** *numbers defined in Eqs. (13)–(16) below. The arrays $M \pm \delta_{ij}$ and $M \pm \delta'_{ij}$ are obtained from M by replacing m_{ij} and m'_{ij} by $m_{ij} \pm 1$ and $m'_{ij} \pm 1$, respectively. Moreover, if an array N is not a pattern, then we assume that $\zeta_N = 0$.*

Define $l_{ij} = -m_{ij} - i - 1 + j$ and $l'_{ij} = -m'_{ij} - i - 1 + j$. If $\mathbf{r} = (r_1, \ldots, r_k) \in \mathbb{Z}^k$ and $t \in \mathbb{Z}$, then we define $\Pi(\mathbf{r}, t) := (-1)^k \prod_{j=1}^{k}(r_j + k + 1 - j + t)$ and $\Pi_i(\mathbf{r}, t) := (-1)^k \prod_{j=1, j\neq i}^{k}(r_j + k + 1 - j + t)$. Then

$$A_{ki}(M) := (\Pi_i(\mathbf{r}'_k(M), l'_{ki}))^{-1}, \tag{13}$$

$$B_{ki}(M) := 4A_{ki}(M)l'_{ki}\Pi(\mathbf{r}_k(M), l'_{ki})\Pi(\mathbf{r}_{k-1}(M), l'_{ki}), \tag{14}$$

$$C_{ki}(M)^{-1} := 2(-1)^k l_{k-1,i}\Pi_i(\mathbf{r}_{k-1}(M), l'_{k-1,i})\Pi_i(\mathbf{r}_{k-1}(M), -l'_{k-1,i}), \tag{15}$$

$$\begin{aligned} D_{kijm}(M) :=& \\ & -A_{ki}(M)A_{k-1,m}(M)C_{kj}(M)\Pi_i(\mathbf{r}'_k(M), l_{k-1,j})\Pi_i(\mathbf{r}'_k(M), -l_{k-1,j} - 1) \\ & \times \Pi_m(\mathbf{r}'_{k-1}(M), l_{k-1,j})\Pi_m(\mathbf{r}'_{k-1}(M), -l_{k-1,j} - 1). \end{aligned} \tag{16}$$

4.4. It is well known that

$$\mathbb{C}^{2p} \otimes \tau_p^\lambda = \sum_{a=1}^{p} \tau_p^{\lambda+\varepsilon_a} + \sum_{a=1}^{p} \tau_p^{\lambda-\varepsilon_a}$$

where $\varepsilon_a = (\overbrace{0, \ldots, 0, 1}^{a}, 0, \ldots, 0)$. Here $\tau_p^{\lambda\pm\varepsilon_a} = 0$ if $\lambda \pm \varepsilon_a$ is not a highest weight. For $1 \leq a \leq p$, set $\lambda' = \lambda + \varepsilon_a$ or $\lambda' = \lambda - \varepsilon_a$ and let

$$p_{\lambda'} : \mathbb{C}^{2p} \otimes \tau_p^\lambda \to \tau_p^{\lambda'} \tag{17}$$

denote the canonical $\mathfrak{sp}(p)$ projection.

4.5. Let $\{e_i \colon i = -p, -p+1, \ldots, -1, 1, \ldots, p\}$ be the standard basis for $\mathbb{C}^{2p}$. We shall identify $\mathbb{C}^{2q} = \operatorname{span}\{e_i \colon i = -p, \ldots, -p+q-1, p-q+1, \ldots, p\}$. Thus $\mathbb{C}^{2q}$ is the subspace of $\mathfrak{sp}(p-q)$-invariants in $\mathbb{C}^{2p}$.

Let $\lambda \in \Lambda^+(p)$ and $\mu \in \Lambda^+(p-q)$. Let v_μ denote a highest weight vector of τ^μ_{p-q}. Suppose that τ^μ_{p-q} occurs in τ^λ_p. Recall (8) that $(\tau^\lambda_p)_{v_\mu}$ denotes the subspace of all $\mathfrak{sp}(p-q)$ highest weight vectors with highest weight μ. Note that this is a module for $\mathfrak{sp}(q)$. A basis of $(\tau^\lambda_p)_{v_\mu}$ is given by

$$\mathcal{B}(\lambda, M_\mu) := \{\zeta_M \in \mathcal{B}(\lambda)\colon d_{p-q}(M) = M_\mu\}.$$

Clearly $\mathbb{C}^{2q} \otimes (\tau^\lambda_p)_{v_\mu} \subset \mathbb{C}^{2p} \otimes \tau^\lambda_p$.

LEMMA 4.5.1: *Suppose that $\lambda \in \Lambda^+(p)$, $\mu \in \Lambda^+(p-q)$ and $\lambda' = \lambda \pm \varepsilon_a$ where $1 \le a \le p$. If $\lambda' \in \Lambda^+(p)$, and τ^μ_{p-q} occurs in both τ^λ_p and $\tau^{\lambda'}_p$, then the restriction of the projection map $p_{\lambda'}$ to the subspace $\mathbb{C}^{2q} \otimes (\tau^\lambda_p)_{v_\mu}$ is nonzero and its image lies in $(\tau^\lambda_p)_{v_\mu}$.*

4.6. PROOF OF LEMMA 4.5.1. The last assertion about the image is clear. We shall prove that the restriction map is nonzero.

Recall that $\mathfrak{sp}(p+1)$ is spanned by $\{F_{ij}\colon i,j = -p-1, -p, \ldots, -1, 1, \ldots, p+1\}$. Let $\mathfrak{sp}(p)$ be embedded in $\mathfrak{sp}(p+1)$ as described in §4.1. Let W denote the span of $\{F_{i,-p-1}\colon i = -p, \ldots, -1, 1, \ldots, p\}$. Then W is a $\mathfrak{sp}(p)$ module under the adjoint action, and $W \simeq \mathbb{C}^{2p}$. Under this isomorphism, $e_p \in \mathbb{C}^{2p}$ (see §4.5) is identified with $F_{p,-p-1}$.

We will first deal with the case $\lambda' = \lambda + \varepsilon_a$. Let $\widetilde{\lambda} = (\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{p+1})$ where $\widetilde{\lambda}_1 = \lambda_1 + 1$ and $\widetilde{\lambda}_{i+1} = \lambda_i$ if $i \ge 1$. Consider the representation $\tau^{\widetilde{\lambda}}_{p+1}$ of $\mathfrak{sp}(p+1)$. We identify τ^λ_p with the subspace

$$\operatorname{span}\{\zeta_{\widetilde{M}} \in \mathcal{B}(\widetilde{\lambda})\colon \mathbf{r}'_{p+1}(\widetilde{M}) = \widetilde{\lambda},\ \mathbf{r}_p(\widetilde{M}) = \lambda\}$$

of $\tau^{\widetilde{\lambda}}_{p+1}$. Let $L\colon W \otimes \tau^\lambda_p \to \tau^{\widetilde{\lambda}}_{p+1}$ be the $\mathfrak{sp}(p)$ map given by Lie algebra action

$$L(X \otimes v) = X.v \quad (X \in W,\ v \in \tau^\lambda_p).$$

Let $p'_a\colon \tau^{\widetilde{\lambda}}_{p+1} \to \tau^{\lambda'}_p$ be the $\mathfrak{sp}(p)$ projection given by

$$p'_a(\zeta_{\widetilde{M}}) = \begin{cases} \zeta_{d_p(\widetilde{M})} & \text{if } \mathbf{r}'_{p+1}(\widetilde{M}) = \widetilde{\lambda},\ \mathbf{r}_p(\widetilde{M}) = \lambda + \varepsilon_a, \\ 0 & \text{otherwise.} \end{cases}$$

By (12), $(p'_a \circ L)(F_{p,-p-1} \otimes \zeta_{\widetilde{M}}) = C_{ka}(\widetilde{M})\zeta_{d_p(\widetilde{M})+\delta_{pa}}$. By Lemma 4.6.1 below, we can find a $\zeta_{\widetilde{M}} \in \mathcal{B}(\lambda, \widetilde{M}_\mu)$ such that $\zeta_{\widetilde{M}+\delta_{pa}} \neq 0$. Hence $p'_a \circ L \neq 0$ and we can identify $p_{\lambda'}$ with $p'_a \circ L$. Under this identification, $F_{p,-p-1} \otimes \zeta_{\widetilde{M}}$ corresponds to an element in $\mathbb{C}^{2q} \otimes (\tau^\lambda_p)_{v_\mu}$ and this proves Lemma 4.5.1 for the case $\lambda' = \lambda + \varepsilon_a$.

We shall adopt the following convention. If $\eta = (\eta_1, \dots, \eta_r) \in \Lambda^+(r)$, then we set

$$\eta_i = \begin{cases} \infty & i \le 0, \\ 0, & i \ge r+1. \end{cases}$$

Lemma 4.6.1: *Define a pattern $\widetilde{M}$ associated with $\widetilde{\lambda}$ as follows:*

(i) $d_{p-q}(\widetilde{M}) = M_\mu$, $\mathbf{r}_{p+1}(\widetilde{M}) = \mathbf{r}'_{p+1}(\widetilde{M}) = \widetilde{\lambda}$ *and* $\mathbf{r}_p(\widetilde{M}) = \lambda$.

(ii) $m_{ij} = \min(\lambda_j, \mu_{2(p-q-i)+j})$ *for* $p-q+1 \le i \le p-1$, $1 \le j \le i$.

(iii) $m'_{ij} = \min(\lambda_j, \mu_{2(p-q-i)+j+1})$ *for* $p-q+1 \le i \le p$, $1 \le j \le i$.

Then $\widetilde{M}$ and $\widetilde{M} + \delta_{pa}$ are patterns associated with $\widetilde{\lambda}$.

Proof: It is easy to see that $\widetilde{M}$ is a pattern. To show that $\widetilde{M} + \delta_{pa}$ is a pattern, it suffices to show that $m'_{p,a-1} \ge \lambda_a + 1$. Indeed, $m'_{p,a-1} = \min(\lambda_{a-1}, \mu_{a-2q})$. Since $\lambda + \varepsilon_a \in \Lambda^+(p)$, $\lambda_{a-1} \ge \lambda_a + 1$. On the other hand, since τ^μ_{p-q} occurs in $\tau_p^{\lambda+\varepsilon_a}$, $\mu_{a-2q} \ge \lambda_a + 1$. Thus $m'_{p,a-1} = \min(\lambda_{a-1}, \mu_{a-2q}) \ge \lambda_a + 1$. ∎

Next we consider the case $\lambda' = \lambda - \varepsilon_a$. We only need to modify the proof for $\lambda + \varepsilon_a$ slightly as follows. First we replace $\widetilde{\lambda}$ by $\widetilde{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_p, 0)$. We identify τ_p^λ with the subspace

$$\operatorname{span}\{\zeta_{\widetilde{M}} \in \mathcal{B}(\widetilde{\lambda}) \colon \mathbf{r}'_{p+1}(\widetilde{M}) = \widetilde{\lambda},\ \mathbf{r}_p(\widetilde{M}) = \lambda\}$$

of $\tau^{\widetilde{\lambda}}_{p+1}$. Define $p'_a \colon \tau^{\widetilde{\lambda}}_{p+1} \to \tau_p^{\lambda'}$ by

$$p'_a(\zeta_{\widetilde{M}}) = \begin{cases} \zeta_{d_p(\widetilde{M})} & \text{if } \mathbf{r}'_{p+1}(\widetilde{M}) = \widetilde{\lambda} - \varepsilon_a,\ \mathbf{r}_p(\widetilde{M}) = \lambda - \varepsilon_a, \\ 0 & \text{otherwise.} \end{cases}$$

The proof proceeds as before and eventually it reduces to Lemma 4.6.2 below (cf. Lemma 4.6.1) and completes the proof of Lemma 4.5.1.

Lemma 4.6.2: *Define a pattern $\widetilde{M}$ as follows:*

(i) $d_{p-q}(\widetilde{M}) = M_\mu$, $\mathbf{r}_{p+1}(\widetilde{M}) = \mathbf{r}'_{p+1}(\widetilde{M}) = \widetilde{\lambda}$ *and* $\mathbf{r}_p(\widetilde{M}) = \lambda$.

(ii) $m_{ij} = \max(\lambda_{2p-2i+j}, \mu_j)$ *and* $m'_{ij} = \max(\lambda_{2p-2i+j+1}, \mu_j)$ *for* $p-q+1 \le i \le p-1$, $1 \le j \le i$.

(iii) $m'_{p,j} = \min(\lambda_j, m_{p-1,j-1})$ *for* $j \ne a-1$ *and* $m'_{p,a-1} = \max(\lambda_a, m_{p-1,a-1})$.

Then there exists $k \in \{a-1, a\}$ such that $\widetilde{M} - \delta'_{p+1,a} - \delta_{pa} - \delta'_{p-1,k}$ is a pattern associated with $\widetilde{\lambda}$.

Proof: We need to consider two cases:

Case 1: If $m_{p-1,a-1} < m_{p,a}$, then $m'_{p,a-1} = \lambda_a$ and $m'_{p,a} = m_{p-1,a-1}$, so that $\widetilde{M} - \delta'_{p+1,a} - \delta_{pa} - \delta'_{p-1,a-1}$ is a pattern associated with $\widetilde{\lambda}$.

CASE 2: If $m_{p-1,a-1} \geq m_{p,a}$, then $m'_{p,a-1} = m_{p-1,a-1}$ and $m'_{p,a} = \lambda_a$. Since $\lambda - \varepsilon_a$ is dominant, $m_{p,a} - 1 = \lambda_a - 1 \geq \lambda_{a+1} \geq m'_{p,a+1} = \min(\lambda_{a+1}, m_{p-1,a})$. Thus $\widetilde{M} - \delta'_{p+1,a} - \delta_{pa} - \delta'_{p-1,a}$ is a pattern associated with $\widetilde{\lambda}$. This completes the proofs of both Lemmas 4.6.2 and 4.5.1. ∎

COROLLARY 4.6.3: *By scaling the projection $p_{\lambda\pm\varepsilon_a}$ if necessary, we have the following formula:*

$$p_{\lambda+\varepsilon_a}(e_p \otimes \zeta_M) = \zeta_{M+\delta_{pa}}, \tag{18}$$

$$p_{\lambda-\varepsilon_a}(e_p \otimes \zeta_M) = \sum_{m=1}^{p} D_{am}(M)\zeta_{M-\delta_{pa}-\delta'_{pm}}, \tag{19}$$

where $\zeta_M \in \tau_p^\lambda$ and $D_{am}(M)$ are **nonzero** *scalars.*

Proof: We have identified $p_{\lambda+\varepsilon_a}$ with $p'_a \circ L$ and e_p with $F_{p,-p-1}$ in the proof of Lemma 4.5.1. By (12), $p'_a \circ L(F_{p,-p-1} \otimes \zeta_M) = C_{p+1,a}(\tilde{M})\zeta_{M+\delta_{pa}}$. By (15), the coefficient $C_{p+1,a}(\tilde{M})$ only depends on $\lambda + \varepsilon_a$ and it disappears if we replace $p_{\lambda+\varepsilon_a}$ by $C_{p+1,a}(\tilde{M})^{-1}p_{\lambda+\varepsilon_a}$. This proves (18). The proof for (19) is similar. ∎

5. A basis of $I_p(s)$

5.1. In this section, we shall use the results of Molev ([Mo]) to construct bases for $I_p(s)$, $I_{p,q}(s,\mu)$ and their subquotients. Recall that $I_p(s)$ has $\tilde{K}$-types ($\tilde{K} = \mathrm{Sp}(p) \times \mathrm{Sp}(p)$)

$$I_p(s) = \sum_{\lambda \in \Lambda^+(p)} V_\lambda, \tag{20}$$

where for each $\lambda \in \Lambda^+(p)$, $V_\lambda \cong \tau_p^\lambda \boxtimes \tau_p^\lambda$. By Proposition 4.3.1, V_λ has a basis

$$\mathcal{B}(V_\lambda) = \{\zeta_M \boxtimes \zeta_{M'}\colon M \text{ and } M' \text{ are patterns associated with } \lambda\}.$$

Thus $\bigcup_{\lambda\in\Lambda^+(p)} \mathcal{B}(V_\lambda)$ is a basis of $I_p(s)$.

5.2. Recall the definition of $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ and $T_{\lambda,\lambda'}$ in §2.4. As a representation of $\tilde{K} = \mathrm{Sp}(p) \times \mathrm{Sp}(p)$, $\tilde{\mathfrak{p}}_{\mathbb{C}} \simeq \mathbb{C}^{2p} \boxtimes \mathbb{C}^{2p}$. We define $F_0 \in \tilde{\mathfrak{p}}_{\mathbb{C}}$ to be the vector corresponding to the vector $e_p \boxtimes e_p \in \mathbb{C}^{2p} \boxtimes \mathbb{C}^{2p}$. Lemma 5.2.1 below describes the Lie algebra action of F_0 on a basis vector $\zeta_M \boxtimes \zeta_{M'}$ in $(\sigma_s, I_p(s))$. We will need this lemma in §7.

LEMMA 5.2.1: *We have*

$$\sigma_s(F_0)(\zeta_M \boxtimes \zeta_{M'}) = \sum_{i=1}^{p} t_{\lambda,\lambda+\varepsilon_i}\Big(\lambda_i + s + p - i + \frac{3}{2}\Big)(\zeta_{M+\delta_{pi}} \boxtimes \zeta_{M+\delta_{pi}})$$
$$+ \sum_{j,m,m'=1}^{p} t_{\lambda,\lambda-\varepsilon_j}\Big(\lambda_i - s + p - j + \frac{1}{2}\Big) D_{jm}(M)$$
$$\times D_{jm'}(M')(\zeta_{M-\delta_{pj}-\delta'_{pm}} \boxtimes \zeta_{M'-\delta_{pj}-\delta'_{pm'}})$$

where $t_{\lambda,\lambda'}$ is a nonzero complex number depending on λ, λ' and s. $D_{am}(M)$ is a nonzero number as given in (19).

Proof: Using Corollary 4.6.3 and §2.4, $T_{\lambda,\lambda'}$ can be interpreted as the composite of the following maps:

(21) $$\tilde{\mathfrak{p}}_{\mathbb{C}} \otimes V_\lambda \overset{t_\lambda}{\simeq} (\mathbb{C}^{2p} \otimes \tau_p^\lambda) \boxtimes (\mathbb{C}^{2p} \otimes \tau_p^\lambda) \overset{p_{\lambda'} \boxtimes p_{\lambda'}}{\longrightarrow} \tau_p^{\lambda'} \boxtimes \tau_p^{\lambda'} \overset{\phi}{\to} \tau_p^{\lambda'} \boxtimes \tau_p^{\lambda'} \overset{t_{\lambda'}}{\simeq} V_{\lambda'}$$

where ϕ is multiplication by $\lambda_i + s + p - a + \frac{3}{2}$ if $\lambda' = \lambda + \varepsilon_a$ and multiplication by $\lambda_i - s + p - a + \frac{1}{2}$ if $\lambda' = \lambda - \varepsilon_a$. The maps t_λ and $t_{\lambda'}$ are isomorphisms uniquely determined up to nonzero scalars. $p_{\lambda'}$ was determined in Corollary 4.6.3. $t_{\lambda,\lambda'}$ is a result of t_λ and $t_{\lambda'}$ and the choice of the canonical bases of V_λ and $V_{\lambda'}$. ∎

By identifying $I_p(s)$ as functions on $\tilde{K}$ (see [J]), one can show that the above lemma holds for all s, that is, $t_{\lambda,\lambda'}$ is independent of s. It would be interesting to determine $t_{\lambda,\lambda'}$ explicitly.

5.3. A BASIS OF R_{v_μ}. Assume that $I_p(s)$ is reducible, and R is an irreducible subquotient of $I_p(s)$. By Theorem 2.4.1, $R = R_t(s)$ for some t. Let $\mu \in \Lambda^+(p-q)$ and let v_μ be a highest weight vector in τ^μ_{p-q}. Recall §3.2 that R_μ denotes the τ^μ_{p-q}-isotypic component of R, and R_{v_μ} denotes the space of $\mathrm{Sp}(p-q)$ highest weight vectors in R_μ.

Since R has $\tilde{K}$-types

$$R = \sum_{\lambda \in \Lambda^+(R)} V_\lambda$$

where

(22) $$\Lambda^+(R) := \{\lambda \in \Lambda^+(p) \colon \lambda_t \geq |s| - p + t - 1/2 \geq \lambda_{t+1}\},$$

it has a basis given by

(23) $$\mathcal{B}(R) = \bigcup_{\lambda \in \Lambda^+(R)} \mathcal{B}(V_\lambda).$$

Under the action of $K = \mathrm{U}(p) \times \mathrm{U}(q)$,

$$R_{v_\mu} = \sum_{\lambda \in \Lambda^+(R_{v_\mu})} \tau_p^\lambda \boxtimes (\tau_p^\lambda)_{v_\mu},$$

where $\Lambda^+(R_{v_\mu}) = \Lambda^+(p, \mu) \cap \Lambda^+(R)$. Now recall that M_μ is the pattern associated with μ such that ζ_{M_μ} is a highest weight vector in τ^μ_{p-q}. Thus the set

$$\mathcal{B}(R_{v_\mu}) = \{\zeta_M \boxtimes \zeta_{M'} \in \mathcal{B}(R) \colon d_{p-q}(M') = M_\mu\} \tag{24}$$

is a basis for R_{v_μ}. More generally, let L_μ be any pattern associated with μ and consider the space

$$R(L_\mu) = \mathrm{span}\{\zeta_M \boxtimes \zeta_{M'} \in \mathcal{B}(R) \colon d_{p-q}(M') = L_\mu\}. \tag{25}$$

Then clearly $R(L_\mu) \cong R_{v_\mu}$ as infinitesimal $\mathrm{Sp}(p,q)$-modules.

5.4. Proof of Theorem 3.3.1. By (24), $R_{v_\mu} = (R_t(s))_{v_\mu}$ is nonzero if and only if $l_1 \leq t \leq l_2$ where l_1 and l_2 are given in Theorem 3.3.1.

Let R and S be irreducible subquotients of $I_p(s)$ such that R_{v_μ} and S_{v_μ} are nonzero. Suppose $S \to R$ in the module diagram. We claim that $S_{v_\mu} \to R_{v_\mu}$. Indeed $S \to R$ implies that there exists $\tilde{K}$-types V_λ and $V_{\lambda'}$ of S and R respectively such that $\lambda' = \lambda \pm \varepsilon_a$ for some a and $T_{\lambda,\lambda'} \neq 0$ (see §2.4). Next we regard $(V_\lambda)_{v_\mu}$ and $(V_{\lambda'})_{v_\mu}$ as K-types of $I_{p,q}(s,\mu)$. Applying Lemma 4.5.1 to (21) shows that the image of $\mathfrak{p} \otimes (V_\lambda)_{v_\mu}$ under $T_{\lambda,\lambda'}$ is nonzero and it lies in $(V_{\lambda'})_{v_\mu}$. This proves the claim and the theorem. ∎

5.5. In the remaining part of this section, we shall establish several lemmas which are needed in the proof of Theorem 1.2.1. If $1 \leq r \leq p-1$ and $\eta \in \Lambda^+(r)$, then R_η shall denote the τ_r^η-isotypic component of R under the action of $\mathrm{Sp}(r) \subset \mathrm{Sp}(q)$.

Lemma 5.5.1: *Let $1 \leq r \leq p-1$, $\eta \in \Lambda^+(r)$, and L_η a pattern associated with η. If $R_\eta \neq 0$, then*

$$R_\eta \cong R(L_\eta) \boxtimes \tau_r^\eta$$

as infinitesimal $\mathrm{Sp}(p, p-r) \times \mathrm{Sp}(r)$-modules. In particular, R_η is irreducible as an infinitesimal $\mathrm{Sp}(p, p-r) \times \mathrm{Sp}(r)$-module if and only if $R(L_\eta)$ is irreducible as an infinitesimal $\mathrm{Sp}(p, p-r)$-module.

Proof: We shall construct an isomorphism $\varphi\colon R_\eta \to R(L_\eta) \boxtimes \tau_r^\eta$ as follows. Let $\zeta_M \boxtimes \zeta_{M'}$ be an element of $\mathcal{B}(R_\eta)$, and let M'' be the pattern obtained by replacing the lowest $2r$ rows of M' by L_η. Then define

$$\varphi(\zeta_M \boxtimes \zeta_{M'}) = (\zeta_M \boxtimes \zeta_{M''}) \boxtimes \zeta_{d_r(M')}. \qquad \blacksquare$$

5.6. THE ACTION OF $\mathfrak{sp}(p, q-r)$. Let R_{v_μ} as above and let A be an infinitesimal $\mathrm{Sp}(p,q)$-submodule of R_{v_μ}. For $p-q+1 \le r \le p$ and $\alpha \in \Lambda^+(r)$, we define

$$A_\alpha = A \cap R_\alpha.$$

LEMMA 5.6.1: *For each $\alpha \in \Lambda^+(r)$, A_α is an infinitesimal* $\mathrm{Sp}(p,p-r) \times \mathrm{Sp}(q-p+r)$ *module. Moreover*

$$A = \bigoplus_{\alpha \in \Lambda^+(r)} A_\alpha. \tag{26}$$

Proof: Let χ denote the infinitesimal character of center of the universal algebra of $\mathfrak{sp}(p,p-r)_{\mathbb{C}}$ acting on $I_{p,p-r}(s,\alpha)$. Let p_χ denote the projection from R onto its χ isotypic component. By Proposition 2.5.1, p_χ on R is the same as the projection onto the τ_r^α-isotypic component R_α of R. This proves (26). Since p_χ commutes with the action of $\mathfrak{sp}(p,p-r) \oplus \mathfrak{sp}(q-p+r)$, A_α is an infinitesimal $\mathrm{Sp}(p,p-r) \times \mathrm{Sp}(q-p+r)$-module. $\blacksquare$

5.7. EIGENSPACES FOR F_{ii}. For $p-q+1 \le i \le p$, let

$$\mathfrak{s}_i = \mathrm{span}\{F_{ii}, F_{i,-i}, F_{-i,i}\} \subseteq \mathfrak{sp}(q)_{\mathbb{C}} \subseteq \mathfrak{sp}(p,q)_{\mathbb{C}}. \tag{27}$$

Then $\mathfrak{s}_i \cong \mathfrak{sl}(2,\mathbb{C})$ and each element of the basis $\mathcal{B}(R)$ for R is an eigenvector for F_{ii}. Indeed by (9),

$$\sigma_s(F_{ii})(\zeta_M \boxtimes \zeta_{M'}) = (|\mathbf{r}_i(M')| + |\mathbf{r}_{i-1}(M')| - 2|\mathbf{r}'_i(M')|)(\zeta_M \boxtimes \zeta_{M'}).$$

PROPOSITION 5.7.1: *Suppose that $a = \sum_{k=1}^m c_k \zeta_{M_k} \boxtimes \zeta_{M'_k}$ is an element of A. If $1 \le i \le m$, and*

$$C_i = \{k\colon 1 \le k \le m,\ \mathbf{r}_j(M'_k) = \mathbf{r}_j(M'_i),\ |\mathbf{r}'_j(M'_k)| = |\mathbf{r}'_j(M'_i)|\ \forall p-q+1 \le j \le p\},$$

then the "subsum"

$$\sum_{k \in C_i} c_k \zeta_{M_k} \boxtimes \zeta_{M'_k} \tag{28}$$

also belongs to A.

Proof: For $p-q+1 \le j \le p$, set $\alpha^j := \mathbf{r}_j(M'_i) \in \Lambda^+(j)$. By (26) we can define the projection $\pi_{\alpha^j}\colon A \to A_{\alpha^j}$. Then

$$\pi_{\alpha^{p-q+1}} \pi_{\alpha^{p-q+2}} \cdots \pi_{\alpha^p}(a) = \sum_{k \in B_i} c_k \zeta_{M_k} \boxtimes \zeta_{M'_k} \in A \tag{29}$$

where

$$B_i = \{k\colon 1 \le k \le m,\ \mathbf{r}_j(M_i') = \mathbf{r}_j(M_k'),\ \forall p-q+1 \le j \le p\}.$$

Then (28) is the image of the projection of the vector in (29) into the simultaneous eigenspaces of $\{F_{jj}\colon j = p-q+1, \dots, p\}$. ∎

6. Proof of Theorem 1.2.1

6.1. We shall prove Theorem 1.2.1 by an induction on q. The case $q = 1$ follows from the results of [BK] but we will sketch an alternative proof. These will be discussed in §7. In this section, we shall assume that the theorem is true for $q - 1 \ge 1$ and will prove that it is also true for q.

For the ease of explaining we will assume the more interesting case where $s + \frac{1}{2} \in \mathbb{Z}$. By Theorem 2.4.1, $I_p(s)$ is reducible unless $s = \pm\frac{1}{2}$. When $s + \frac{1}{2} \notin \mathbb{Z}$, Theorem 2.4.1 states that $I_p(s)$ is irreducible and the proof of Theorem 1.2.1 for this case is similar but easier. We will leave it to the reader.

6.2. Notation. Throughout this section, $s + \frac{1}{2} \in \mathbb{Z}$ and $R = R_t(s)$ will denote an irreducible subquotient of $I_p(s)$ (cf. (4)). Let $\mu \in \Lambda^+(p-q)$ and let M_μ denote the pattern which corresponds to the highest weight vector in τ^{μ}_{p-q}. We assume that the subquotient $R' = R_{v_\mu}$ of $I_{p,q}(s,\mu)$ is nonzero.

6.3. First we state a crucial lemma.

Main Lemma: *Let R and R_{v_μ} as above. Then there exist*

(i) $\theta \in \Lambda^+(p-q+1)$,

(ii) $\lambda \in \Lambda^+(p)$,

(iii) *a pattern $M^\theta = (m^\theta_{ij}, {m^\theta_{ij}}')$ associated with λ which satisfies $\mathbf{r}_{p-q+1}(M^\theta) = \theta$, $d_{p-q}(M^\theta) = M_\mu$, ${m^\theta_{p-q+1,j}}' = \max(\mu_j, \theta_{j+1})$,*

(iv) *$w_0 := v_\lambda \boxtimes \zeta_{M^\theta} \in R_{v_\mu}$ where v_λ is a highest weight vector in τ^λ_p, such that every nonzero infinitesimal Sp(p,q)-submodule of R_{v_μ} contains the vector w_0.*

Consequently, w_0 generates a unique irreducible infinitesimal Sp(p,q)-submodule U in R_{v_μ}. Moreover, if s is replaced by $-s$, then the conclusion of the lemma remains true with the same choice of $(\theta, \lambda, M^\theta)$.

We will now assume the Main Lemma and proceed to prove Theorem 1.2.1. The proof of the Main Lemma will occupy §6.4 to §6.7.

LEMMA 6.3.1: *Let $L_\theta = d_{p-q+1}(M^\theta)$. Then U contains (see (25))*

$$R(L_\theta) = \operatorname{span}\{\zeta_M \boxtimes \zeta_{M'} \in \mathcal{B}(R_{v_\mu}) \colon d_{p-q+1}(M') = L_\theta\}.$$

Proof: Note that the vector w_0 in the Main Lemma lies in $R(L_\theta)$. Let v_θ be a highest weight vector in τ^θ_{p-q+1}. Then $R(L_\theta) \simeq R_{v_\theta}$ as infinitesimal $\mathrm{Sp}(p, q-1)$-modules. Thus by induction hypothesis, $R(L_\theta)$ is irreducible and hence it is contained in U. ∎

Proof of Theorem 1.2.1.: Note that $\mathfrak{sp}(p,q)_\mathbb{C}$ contains $\mathfrak{sp}(p,q-1)_\mathbb{C} \times \mathfrak{s}$ where $\mathfrak{s} = \mathfrak{s}_{p-q+1} \simeq \mathfrak{sl}(2,\mathbb{C})$ was defined in (27) by setting $i = p-q+1$. First we claim that $R(L_\theta)$ can be characterized as the subspace in $(R_{v_\mu})_\theta$ such that $\mathfrak{s}$ acts by the highest weight

$$r_0 := \sum_j \theta_j + \sum_j \mu_j - 2|\mathbf{r}'_{p-q+1}(M_\theta)|.$$

Indeed $m^\theta_{p-q+1,i}$ takes the smallest possible value for all j so that $M_\theta - \delta'_{p-q+1,j}$ is no longer a pattern.

Next we consider the dual representation $R^* = (R_t(s))^* = R_t(-s)$. In this way $(R_{v_\mu})^* \simeq (R_t(-s))_{v_\mu}$. By the Main Lemma, the pattern L_θ for $(R_t(-s))_{v_\mu}$ is the same as that for $(R_t(s))_{v_\mu}$. Hence by Lemma 6.3.1, $(R_t(-s))_{v_\mu}$ contains a unique irreducible submodule U' and U' contains the subspace in $(R^*_{v_\mu})_\theta$ where $\mathfrak{s}$ acts by the highest weight r_0 (resp. lowest weight $-r_0$). This implies that R_{v_μ} contains a unique quotient and $R(L_\theta)$ generates R_{v_μ}. This shows that $R_{v_\mu} = U$ is irreducible. ∎

6.4. The remaining section is devoted to the proof of the Main Lemma. Let R_{v_μ} be as in §6.2 and let A be a nonzero infinitesimal $\mathrm{Sp}(p,q)$-submodule of R_{v_μ}.

We will briefly describe the main idea of the proof. First we pick an arbitrary vector

$$w_\theta = \sum_{l=1}^{m} c_l v_\lambda \boxtimes \zeta_{M_l} \in (V_\lambda)_{v_\mu} \subset A \tag{30}$$

where $c_l \in \mathbb{C}$, v_λ is a highest weight vector of τ^λ_p, and $d_{p-q}(M_l) = M_\mu$. We will show that by the actions of the Lie algebra on w_θ, we can replace w_θ successively such that we impose more and more conditions on the patterns M_l. Eventually only one pattern, namely M^θ in the Main Lemma, satisfies all the conditions.

6.1. By Lemma 5.6.1,

$$A = \sum_{\theta \in \Lambda^+(p-q+1)} A_\theta.$$

Pick θ such that $A_\theta \neq 0$. Let N_θ be a pattern associated with θ such that $d_{p-q}(N_\theta) = M_\mu$. Then by Lemma 5.5.1, $R_\theta \cong R(N_\theta) \boxtimes \tau^\theta_{p-q+1}$. By the induction hypothesis, $R(N_\theta)$ is an irreducible infinitesimal Sp$(p, q-1)$-module, so that R_θ is an irreducible infinitesimal Sp$(p, q-1) \times$ Sp$(p-q+1)$-module. Let $\mathfrak{a}$ denote the complexified Lie algebra of Sp$(p-q+1)$. Then we have

$$\mathcal{U}(\mathfrak{a})A_\theta = R_\theta. \tag{31}$$

Define $\lambda = \lambda(\theta)$ by

$$\lambda_i = \begin{cases} \theta_1 + 1 & \text{if } i \le \min(t, 2q-2) \\ \theta_{i-2q+2} & \text{if } s \ge 2q-1 \text{ and } 2q-1 \le i \le t \\ \min(\ell_t, \theta_{i-2q+2}) & \text{if } i > t \end{cases} \tag{32}$$

where

$$\ell_t = |s| - p + t - 1/2.$$

R_θ contains $(V_\lambda)_\theta$ and this in turn contains a vector of the form

$$w = v_\lambda \boxtimes \zeta_M \tag{33}$$

where v_λ is a highest weight vector in τ_p^λ and $M = (m_{ij}, m'_{ij})$ is a pattern with the following properties:

(A1) M is a associated with λ, that is, $\mathbf{r}_p(M) = \lambda$,

(A2) $\mathbf{r}_{p-q+1}(M) = \theta$,

(A3) $d_{p-q}(M) = M_\mu$,

(A4) the entries between $r_p(M)$ and $r_{p-q+1}(M)$ are given the largest possible values. Specifically, for $p-q+2 \le i \le p-q$ and $1 \le j \le i$, we set

$$m_{ij} = \min(\lambda_j, \theta_{2p-2q-2i+j+2}), \quad m'_{ij} = \min(\lambda_j, \theta_{2p-2q-2i+j+1}).$$

Lemma 6.5.1: *Let R and A be as in §6.4. If $\theta \in \Lambda^+(p-q+1)$ is such that $A_\theta \neq 0$, then A_θ contains a vector w_θ of the form*

$$w_\theta = \sum_l c_l v_\lambda \boxtimes \zeta_{M_l}$$

where $\lambda = \lambda(\theta) \in \Lambda^+(p)$ is defined in (32) and all M_l are patterns satisfying the conditions (A1)–(A4).

Proof: Recall that $\mathcal{U}(\mathfrak{a})A_\theta = R_\theta$ and it contains w in (33). Hence there exist $X_i \in \mathcal{U}(\mathfrak{a})$ and $u_i \in A_\theta$ of the form given in Proposition 5.7.1 such that

$$\sum_i X_i u_i = w = v_\lambda \boxtimes \zeta_M.$$

We also recall (24) that $\mathcal{B}(R_{v_\mu})$ is a basis for R_{v_μ} and $A_\theta \subseteq A \subseteq R_{v_\mu}$. Thus each u_i is a linear combination of basis vectors from $\mathcal{B}(R_{v_\mu})$. Since the action of X_i on the basis vectors of τ_p^λ only affects the lowest $2p-2q+1$ rows of the patterns, at least one of the u_i, say u_1, is of the form

$$u_1 = \sum_{l=1}^{m} c_l v_\lambda \boxtimes \zeta_{M_l} \tag{34}$$

where for each l, $c_l \in \mathbb{C}$, M_l is a pattern associated with λ such that

$$\mathbf{r}_j(M_l) = \mathbf{r}_j(M), \quad |\mathbf{r}'_j(M_l)| = |\mathbf{r}'_j(M)|$$

for $p-q+2 \le j \le p$. Since M satisfies (A4), the entries m_{ij} and m'_{ij} between $r_p(M)$ and $r_{p-q+1}(M)$ are given the largest possible values. It follows that if $M_l = (p_{ij}, p'_{ij})$, then

$$p_{ij} = m_{ij} \quad \text{and} \quad p'_{ij} \le m'_{ij}$$

for $p-q+2 \le i \le p$ and $1 \le j \le i$. On the other hand, for each $p-q+2 \le i \le p$, we have

$$|\mathbf{r}_i(M_l)| = \sum_{j=1}^{i} p'_{ij} = \sum_{j=1}^{i} m'_{ij} = |\mathbf{r}_i(M)|.$$

Hence $p'_{ij} = m'_{ij}$ for all $p-q+2 \le i \le p$ and $1 \le j \le i$. This proves the lemma. ∎

Lemma 6.5.2: *Let R and A be as in §6.4. Let k be an integer satisfying $1 \le k \le p-q+1$. Suppose that the following conditions hold:*

(i) $A_\theta \neq 0$.
(ii) *If $k \ge 2$, then $\min(\theta_{k-1}, \mu_{k-2}) > \theta_k$.*
(iii) *If $k = t+1 \ge 1$ then $\theta_{t+1} < l_t$.*

Then $A_{\theta+\varepsilon_k} \neq 0$.

Proof: Let

$$w_\theta = \sum_l c_l v_\lambda \boxtimes \zeta_{M_l}$$

be the vector in A_θ as described in Lemma 6.5.1. We consider two cases.

CASE 1: Suppose for some l, $M_l = (a_{ij}, a'_{ij})$ is such that

$$a'_{p-q+1,k-1} > \theta_k.$$

Note that this is trivially satisfied if $k = 1$. Then $M_l + \delta_{p-q+1,k}$ is a pattern associated with λ and $\mathbf{r}_{p-q+1}(M_l + \delta_{p-q+1,k}) = \theta + \varepsilon_k$. Let $F = F_{p-q+1,-(p-q+2)}$. Since $F \in \mathfrak{sp}(p,q)_{\mathbb{C}}$, $F(w_\theta) \in A$. By Proposition 4.3.1, $F(w_\theta)$ has a nonzero component

$$c_l v_\lambda \boxtimes \zeta_{M_l + \delta_{p-q+1,k}}.$$

Thus $\pi_{\theta+\varepsilon_k}(F(w_\theta)) \neq 0$ and $A_{\theta+\varepsilon_k} \neq 0$.

CASE 2: Suppose for each l, the pattern $M_l = (a_{ij}, a'_{ij})$ is such that

$$a'_{p-q+1,k-1} = \theta_k.$$

Let $Y = F_{-(p-q+1),p-q+1}$. Then by Proposition 4.3.1, $Y(\zeta_{M_l})$ contains a nonzero component in $\zeta_{M_l+\delta'_{p-q+1,k-1}}$. Now for the pattern $M_l + \delta'_{p-q+1,k-1}$, note that

$$a'_{p-q+1,k-1} + 1 > \theta_k.$$

Thus we replace the vector w_θ by $Y(w_\theta)$ and proceed as in Case 1.

We need condition (iii) because $R_\theta \neq 0$ imply that $\theta_{t+1} \leq l_t$. ∎

6.6. Recall Eq. (26) that

$$A = \sum_{\theta \in \Lambda^+(p-q+1)} A_\theta.$$

We define

$$(\theta_1^{\min})_A = \min\{\theta_1 \colon A_\theta \neq 0\}.$$

Now a theorem of Harish-Chandra states that an admissible Harish-Chandra module with a unique infinitesimal character is of finite length (for example, see Cor. 5.4.16 [Vo]). Thus R_{v_μ} has only finitely many distinct submodules, and so we can define

(35)

$$\Theta_1 = \max\{(\theta_1^{\min})_A \colon A \text{ is an infinitesimal } \mathrm{Sp}(p,q)\text{-submodule of } R_{v_\mu} \text{ or } (R_{v_\mu})^*\}.$$

LEMMA 6.6.1: *Let R and A be as in §6.4 and let $\theta \in \Lambda^+(p-q+1)$ be defined by*

(1) $\theta_1 = \theta_2 = \Theta_1$;
(2) $\theta_j = \mu_{j-2}$ *for all* $3 \leq j \leq p-q+1$, $j \neq t+1, t+2$; *and*

(3) $\theta_j = \min(\mu_{j-2}, \ell_t)$ *for* $j = t+1, t+2$.
Then

(i) A_θ *contains a vector of the form* $\tilde{w}_\theta = v_\lambda \boxtimes v_0$ *where*

$$v_0 = \sum_{i=a}^{b} c_i \zeta_{M_i}, \tag{36}$$

$a \le b \le c$ *are nonnegative integers,* $\lambda = \lambda(\theta)$ *and* M_i *is a pattern satisfying the conditions (A1) to (A4), and*

$$\mathbf{r}'_{p-q+1}(M_i) = (\Theta_1, \mu_1, \ldots, \mu_{t-2}, i, \min(\mu_t, \ell_t), \mu_{t+1}, \ldots, \mu_{p-q}, c-i).$$

(ii) *Furthermore, we may assume that* $F_{p-q+1,-(p-q+1)} v_0 = 0$.

Proof: Take any θ such that $\theta_1 = (\theta_1^{\min})_A$ and $A_\theta \neq 0$ and apply Lemma 6.5.2 repeatedly to get (36). It reduces to 3 possible cases, namely, (1) $\mu_t \le \mu_{t-1} \le \ell_t$, (2) $\mu_t \le \ell_t \le \mu_{t-1}$ and (3) $\ell_t \le \mu_t \le \mu_{t-1}$. In the first case $a = b = \mu_{t-1}$, and $\tilde{w} = c_a v_\lambda \boxtimes \zeta_{M_a}$ and $\mathbf{r}'_{p-q+1}(M_a) = (\Theta_1, \mu_1, \ldots, \mu_{t-2}, \mu_{t-1}, \mu_t, \ldots, \mu_{p-q}, c-a)$. In Cases (2) and (3), $\tilde{w}_\theta$ is a sum. We will leave the details to the reader.

Next we prove the last assertion. Let $Y = F_{p-q+1,-(p-q+1)}$ and suppose $Yv_0 \neq 0$. Then we may replace $\tilde{w}_\theta$ by $v_\lambda \boxtimes Yv_0 \in A$ and (i) still holds. In this case c is decreased by 1. Since c is nonnegative, the process must stop after finite iterations and $Yv_0 = 0$. ∎

6.7. Proof of the Main Lemma. Set $r = p-q+1$. Let v_0 be as in (36). By (10),

$$0 = F_{r,-r} v_0 = \sum_{i=a}^{b} c_i A_{rt}(M_i) \zeta_{M_i - \delta'_{rt}} + c_i A_{rr}(M_i) \zeta_{M_i - \delta'_{rr}}. \tag{37}$$

We consider 2 cases.

Case 1: Suppose $a = c$. We set $M^\theta = M_a$, $w_0 = \tilde{w}_\theta = v_\lambda \boxtimes \zeta_{M_a}$ in Lemma 6.6.1 and we are done. In this case $a = b = \max(\mu_t, \theta_{s+1})$ by (37) and

$$\mathbf{r}'_r(M_a) = (\Theta_1, \mu_1, \ldots, \mu_{t-2}, \min(\mu_{t-1}, \ell_t), \min(\mu_t, \ell_t), \ldots, \mu_{p-q}, 0).$$

Case 2: Suppose $a < c$ and we will reduce this to Case 1. By comparing the coefficients of the canonical bases in (37), we conclude that $a = \max(\mu_t, \theta_{t+1})$, $c_a \neq 0$, $M_a - \delta'_{rr}$ is a pattern and the coefficient of $\zeta_{M_a - \delta'_{rr}}$ is

$$c_a A_{rr}(M_a) + c_{a+1} A_{rt}(M_{a+1}) = 0. \tag{38}$$

Let $F = F_{r,-r-1}$ and we need the following lemma.

LEMMA 6.7.1: *Write $F(v_0)$ as a linear combination of the canonical basis vectors of* Sp(p). *Then the coefficient of $\zeta_{M_a-\delta_{rr}-\delta'_{rr}-\delta_{r-1,r-1}}$ is nonzero.*

Proof: By applying (12), we expand Fv_0 in terms of the canonical bases. One can check that the coefficient of the basis vector $\zeta_{M-\delta_{rr}-\delta'_{rr}-\delta_{r-1,r-1}}$ is

(39) $$c_a D_{r,r,r,r-1}(M_a) + c_{a+1} D_{r,r,r,s}(M_{a+1}).$$

We claim that (39) is nonzero. Indeed by (38) it suffices to check that

(40) $$D := \det\begin{pmatrix} A_{rr}(M_a) & A_{rt}(M_{a+1}) \\ D_{r,r,r,p-q}(M_a) & D_{rrrt}(M_{a+1}) \end{pmatrix} \neq 0.$$

We substitute (13) to (16) into the entries of the above determinant. The main observation is that the entries are products of numbers depending on the rows of M_a and M_{a+1}. Moreover, the rows of the M_a and M_{a+1} are the same except at the r'-th row. Therefore, the entries in D have many common factors. Calculation shows that D is a nonzero multiple of the number

(41) $$(l_r - l'_t)(l_r + l'_t + 1) - (l_r - l'_r)(l_r + l'_r + 1) = -l'_t(l'_t + 1) + l'_r(l'_r + 1)$$

where $l_i = -m_{r,i} - i$ and $l'_i = -m'_{r,i} - i$. The right hand side of (41) a nonzero negative integer because $0 > l'_r > l'_t$. This completes the proof of Lemma 6.7.1. ∎

We continue with the proof of the Main Lemma. Let $w_1 \in A$ denote the image of the projection of $F(\tilde{w}_\theta)$ into the $\tau_r^{\theta-\varepsilon_r}$-isotypic component. By Lemma 6.7.1, $w_1 \neq 0$. Then by (12), Fw_1 has a nontrivial τ_r^θ-isotypic component. Now by (11), the vector $F_{-r-1,r+1}(Fw_1)$ has a subsum of the same form as $\tilde{w}_\theta$ in Lemma 6.6.1 except that c is replaced by $c-1$. Thus by applying this procedure repeatedly, we reduce to Case 1.

The assertion about the irreducible submodule U is clear. Finally, we note that the definitions of θ and M^θ depend only on the K-types of $R_{v_\mu} = (R_t(s))_{v_\mu}$. Since the dual representation $(R_{v_\mu})^* \simeq (R_t(-s))_{v_\mu}$ has the same K-types, the lemma holds with same choice of $(\theta, \lambda, M^\theta)$. ∎

7. Proof of Theorem 1.2.1 for q=1

7.1. It remains to prove Theorem 1.2.1 for q=1. We will give two proofs of this fact:

7.2. FIRST PROOF. One can deduce the number of irreducible subquotients of $I_{p,1}(s,\mu)$ from [BK] (also see [HK]). A case by case verification shows that the number agrees with the number of subquotients given by Theorem 1.2.1.

7.3. SECOND PROOF. We will prove the Theorem by 'induction from $q = 0$'. The proof was not included in §6 because certain special considerations will complicate the (already complicated) treatment there. However, the idea is similar so we will only give a sketch of the proof.

First we set $q = 1$ and $\lambda(\theta) = \theta$ throughout §6. In the Main Lemma, part (i) is the same as (ii). In Lemma 6.3.1, $R(L_\theta)$ is just an irreducible K-type of $\mathrm{Sp}(n,1)$ and $(R_{v_\mu})_\theta = (V_{\lambda(\theta)})_{v_\mu}$. The proof in §6.3 works formally for $q = 1$.

It remains to prove the Main Lemma. Conditions (A1) and (A2) in §6.5 are the same. The proof proceeds exactly as that in §6.5 to §6.7 except in the proofs of Lemmas 6.5.2 and 6.7.1 we replace the operator F by the operator $F_0 \in \mathfrak{sp}(n,1)$ in Lemma 5.2.1. This gives the Main Lemma and completes the proof for $q = 1$. ∎

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